

Use of the Cross Wigner-Ville Distribution for Estimation of Instantaneous Frequency

Boualem Boashash and Peter O'Shea

Abstract—This correspondence presents an iterative instantaneous frequency (IF) estimation scheme in which successive IF estimates are obtained from the peak of the cross Wigner-Ville distribution (XWVD) using a reference signal synthesized from an initial IF estimate. Theoretical and practical aspects of performance are discussed, and compared with other methods.

I. INTRODUCTION

The concept of instantaneous frequency (IF) has become very useful in many engineering applications where it is used to describe the time-varying nature of a signal. A thorough tutorial review of this concept is provided in [3].

A number of approaches have been proposed for estimating the IF of a constant amplitude phase modulated signal and were reviewed in [2], [3], [7], and [6]. Perhaps the simplest approach is to use the direct definition, i.e., differentiate the phase of the analytic signal [13]:

$$f_i(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} \quad (1)$$

where $z_s(t) = a(t)e^{j\phi(t)}$ is the analytic signal associated with the real signal $s(t)$. The estimator based on the direct definition is unbiased, but high in variance. For many signals one can obtain a lower variance estimate which is still unbiased by using alternative methods [7].

One approach which gives low variance estimates is based on extracting the peak of time-frequency distributions (TFD's). That is, just as good estimates of frequency may be obtained from the peak of the spectrum, useful estimates of the IF may be obtained from the peak of TFD's. The peak of the Wigner-Ville distribution (WVD), for example, is an optimal IF estimator for linear frequency modulated (FM) signals at high signal-to-noise ratio (SNR) [11]. The bilinear nature of most TFD's, however, accentuates the effects of noise and limits their use for IF estimation at low SNR. For this reason a method based on a linear time-varying spectral representation, the cross Wigner-Ville distribution (XWVD), is proposed in this correspondence.

II. XWVD BASED IF ESTIMATION

The following iterative XWVD based algorithm is proposed for estimating the IF:

- 1) *Initialization*: Obtain an IF estimate and form a unit amplitude signal from it.
- 2) *Estimation Procedure*: Using the current signal estimate as reference, form an XWVD and extract the peak as the new IF estimate.
- 3) *Recursion*: Repeat the procedure from step 1 until the IF estimate differs from the previous one by less than a specified amount.

Manuscript received January 25, 1990; revised January 28, 1992. This work was supported by the Australian Research Council (ARC) and the Australian Defence Research and Technology Organisation (DSTO).

The authors are with the Center for Signal Processing Research, Queensland University of Technology, Brisbane, Australia 4001.

IEEE Log Number 9206000.

Any method can be used to obtain the initial IF estimate. However, the authors have found that STFT peak based estimates tend to result in comparatively rapid convergence at low SNR. At high SNR, the simplest IF estimate to use is one based on the direct definition in (1). Using such an estimate, the reference signal becomes the observed signal and the XWVD scheme reduces to finding the peak of the WVD.

The XWVD scheme exhibits typical nonlinear estimator performance: above a SNR threshold it has very low variance (and in fact meets the Cramer-Rao (CR) bound), but fails badly below the threshold. A consideration of the noise performance is provided in the following sections.

A. Performance of IF Estimation Using the XWVD

Consider an infinite length analytic signal $z_r(t)$, derived from the observation sequence, which belongs to the class of constant amplitude phase modulated signals. In the XWVD based algorithm an XWVD is formed between $z_r(t)$ and a unit amplitude analytic signal, $\hat{z}_s(t)$, synthesized from an initial IF estimate of the signal. This is expressed as

$$W_{z_r \hat{z}_s}(t, f) = \int_{-\infty}^{+\infty} z_r(t + \tau) \hat{z}_s^*(t - \tau) e^{-j4\pi f\tau} d\tau \quad (2)$$

$$= \mathcal{F}_{\tau \rightarrow 2f} \{z_r(t + \tau) \hat{z}_s^*(t - \tau)\} \quad (3)$$

where

$$\mathcal{F}_{\tau \rightarrow 2f}$$

signifies Fourier transformation in the τ variable (with a frequency scaling of a half). $\hat{z}_s(t)$ is given by

$$\hat{z}_s(t) = \exp \left[j2\pi \int_{-\infty}^t \hat{f}_i(t) dt + \hat{\phi}_s \right] \quad (4)$$

where $\hat{f}_i(t)$ is the initial IF estimate, and $\hat{\phi}_s$ is a phase constant.

The initial IF estimate, $\hat{f}_i(t)$, will differ from the true IF by some error term. The IF estimate, then, may be considered to be the sum of a true term and an error term:

$$\hat{f}_i(t) = f_i(t) + f_e(t). \quad (5)$$

This implies that in the time domain, $\hat{z}_s(t)$ may be written as the product of a complex signal reflecting the "true IF" and a complex signal reflecting the "IF error," i.e.,

$$\hat{z}_s(t) = z_s(t) \cdot z_e(t) \quad (6)$$

where $z_e(t) = |z_e| \exp [j2\pi \int_{-\infty}^t f_e(t) dt + \phi_e]$, $|z_e|$ is the amplitude of $z_e(t)$, and ϕ_e is a phase constant. Equation (3) may then be rewritten as

$$\begin{aligned} W_{z_r \hat{z}_s}(t, f) &= \mathcal{F}_{\tau \rightarrow 2f} \{z_r(t + \tau) z_s^*(t - \tau) z_e^*(t - \tau)\} \\ &= \mathcal{F}_{\tau \rightarrow 2f} \{z_r(t + \tau) z_s^*(t - \tau)\} \mathcal{F}_{\tau \rightarrow 2f} \{z_e^*(t - \tau)\} \\ &= W_{z_r z_s}(t, f) \mathcal{F}_{\tau \rightarrow 2f} \{z_e^*(t - \tau)\} \end{aligned} \quad (7)$$

where $\mathcal{F}_{\tau \rightarrow 2f}$ denotes convolution in the frequency variable f . From (7) it can be seen that the XWVD estimate which is formed from the IF estimate is the "true" XWVD smeared by the frequency scaled Fourier transform of the error term.

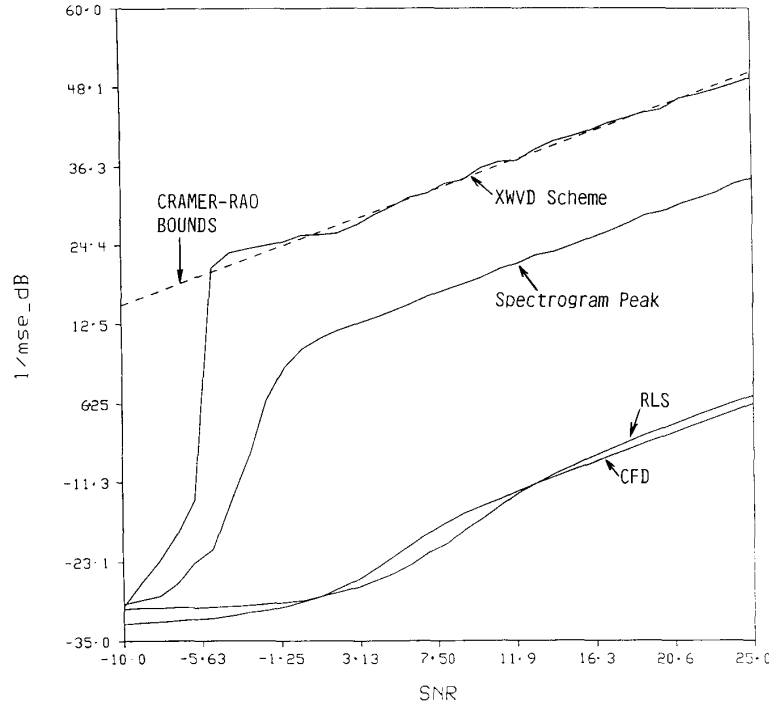


Fig. 1. IF estimate variance versus the Cramér-Rao bound.

B. Convergence Aspects

We first consider the problem of convergence for signals with linear frequency laws. Considerations of how the scheme performs for deviations of the FM law from nonlinearity will be given in a subsequent section.

Analysis shows that if the true signal is a linear FM signal, convergence *will* occur (in the absence of noise) for any arbitrary initial IF estimate. The proof is given below.

Proof: Denote the IF error after the i th iteration as $f_{ei}(t)$ and its corresponding time domain reconstruction as $z_{ei}(t)$. Then the initial IF error $f_{e0}(t)$, is constrained to be between the normalized frequencies $-1/4$ and $+1/4$ (from the Nyquist criterion). That is,

$$|f_{e0}(t)| \leq 1/4 \quad \text{for all } t. \quad (8)$$

Since the initial IF error is constrained to be between $-1/4$ and $+1/4$, a simple superposition argument can be used to show that the peak frequency in the spectrum of $z_{e0}(t)$ is likewise constrained to be between $-1/4$ and $+1/4$. That is,

$$|f_{\text{peak}0}(t)| \leq 1/4 \quad \text{for all } t \quad (9)$$

where $f_{\text{peak}0}(t)$ is the frequency which maximizes

$$\left| \mathcal{F}_{\tau \rightarrow f} \{z_{e0}^*(t - \tau)\} \right|.$$

Since (7) indicates that $f_{e1}(t)$ is the frequency which maximizes

$$\left| \mathcal{F}_{\tau \rightarrow 2f} [z_{e0}^*(t - \tau)] \right|,$$

then

$$|f_{e1}(t)| \leq 1/8 \quad \text{for all } t. \quad (10)$$

Again, since $f_{e1}(t)$ is constrained to be between $-1/8$ and $+1/8$, the peak of the spectrum of $z_{e1}(t)$ will be constrained to the interval

$[1/8, +1/8]$, i.e.,

$$|f_{\text{peak}1}(t)| \leq 1/8 \quad \text{for all } t \quad (11)$$

where $f_{\text{peak}1}(t)$ is the frequency which maximizes

$$\left| \mathcal{F}_{\tau \rightarrow f} \{z_{e1}^*(t - \tau)\} \right|.$$

Again, since (7) implies that $f_{e2}(t)$ is the frequency which maximizes

$$\left| \mathcal{F}_{\tau \rightarrow 2f} [z_{e1}^*(t - \tau)] \right|$$

then

$$|f_{e2}(t)| \leq 1/16 \quad \text{for all } t. \quad (12)$$

By similar reasoning, after the i th iteration, the IF error will be given by

$$|f_{ei}(t)| \leq 2^{-2-i} \quad \text{for all } t \quad (13)$$

and so

$$\lim_{i \rightarrow \infty} |f_{ei}(t)| \rightarrow 0 \quad \text{for all } t. \quad \text{Q.E.D.} \quad (14)$$

Convergence has been shown above to be assured where there is no noise. For the case where noise exists, the XWVD scheme will assuredly converge for asymptotically long signals. This is clear from the fact that the contribution of the observation noise to the XWVD magnitude will tend to a constant level asymptotically if a Blackman-Tukey type spectral estimator is used. Hence the noise will have no effect on estimation of the peak of the XWVD, asymptotically.

In practice, as signal length is reduced, a threshold effect occurs, in much the same way as for conventional frequency estimation.

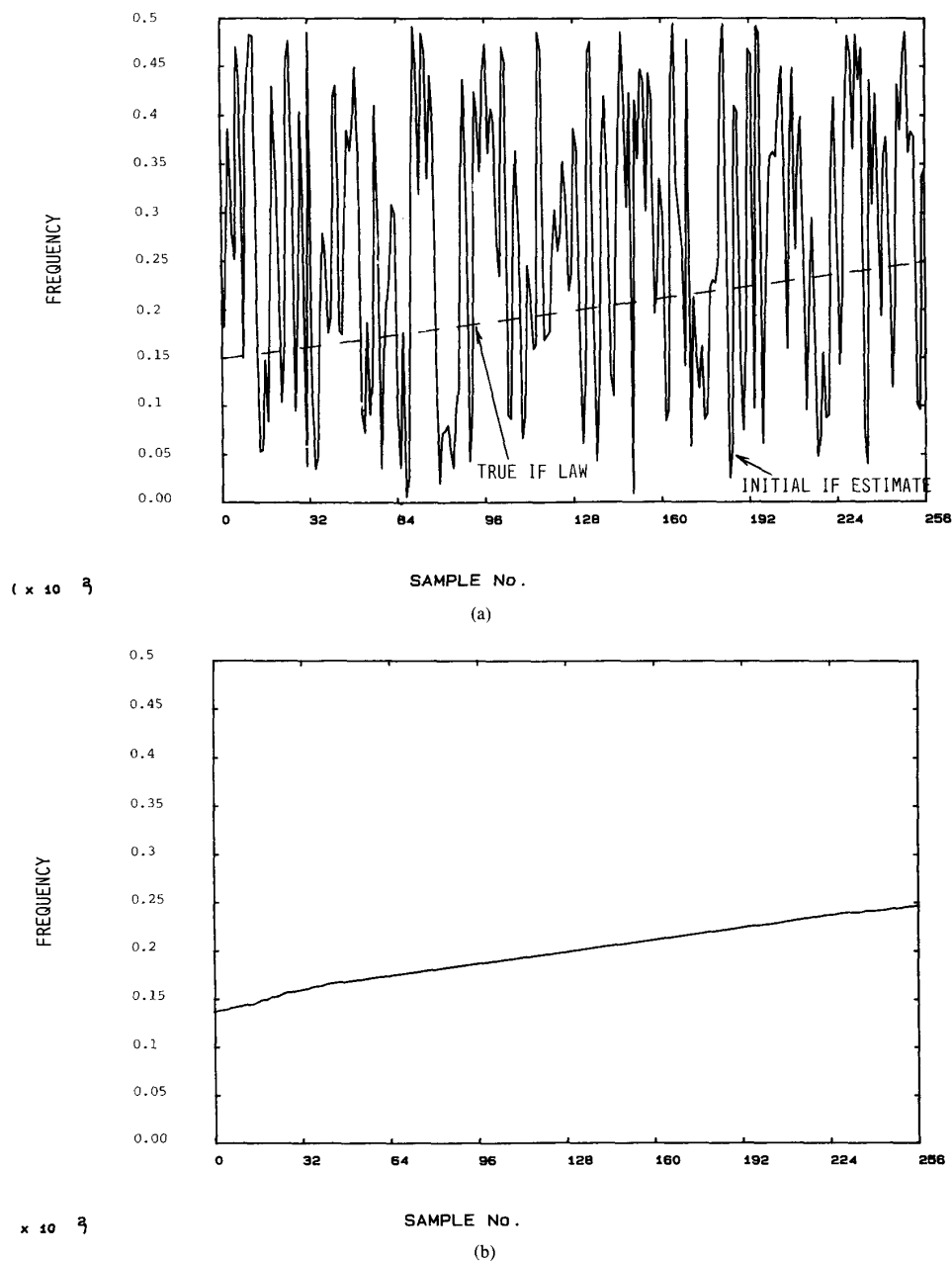


Fig. 2. (a) Initial and true IF estimates; (b) XWVD based IF estimate. Sfreq = 200.0 Hz. $N = 256$.

For real data, the XWVD based scheme is implemented on long signals via a sliding window, the window length being such that the frequency law within the window is very close to linear. Under these conditions, and if an STFT peak starting estimate is used, the SNR threshold has been observed to be almost identical to the threshold for stationary tones analyzed with the same window [12]. Fig. 1 shows simulations of the variance of the XWVD scheme for estimating the IF of a long linear FM signal using a 128 point window. The XWVD estimates are seen to meet the CR bounds above the -4 dB threshold point. The method is seen to compare favorably with other methods. In order to conform to the scenario in

[12] this comparison uses white complex (as opposed to analytic) noise. This fact does not alter the comparative performance of the method. Further comparisons are given in [7] and [3].

C. Computer Simulation

The dashed line of Fig. 2(a) shows the true IF law for signal 1, which is a 256 point linear chirp signal sampled at 1 Hz, and imbedded in -8 dB white Gaussian noise. An initial IF estimate which is significantly biased from the true law is also shown in Fig. 2(a) as a full line. It was obtained by a finite differencing of a white

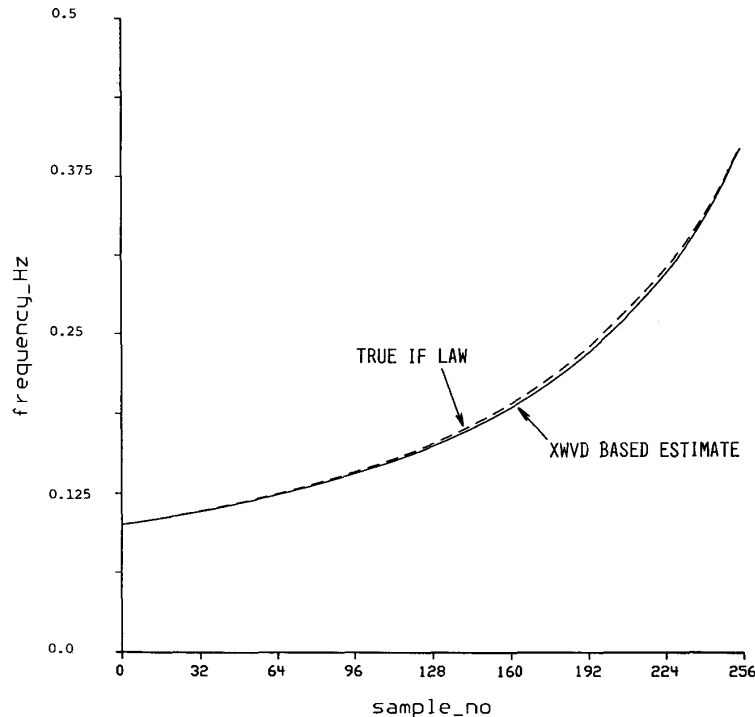


Fig. 3. True IF and XWVD based IF estimate for hyperbolic FM signal.

Gaussian random signal, and thus represents a highly random starting estimate. The XWVD based estimate after ten iterations is displayed in Fig. 2(b). Convergence has occurred (apart from some errors at the endpoints due to the time-limiting problems). End portions of the signal must in fact be determined by linear extrapolation. For signals which are not so noisy, the number of iterations required for convergence is much lower, typically one or two.

D. Extensions

This section briefly discusses the implications of extending the scheme to nonlinear FM signals and to wide-band FM signals. Extensions to multicomponent signals is beyond the scope of this correspondence.

The XWVD based scheme is particularly effective for signals exhibiting a linear frequency law. In fact, asymptotic convergence has only been shown to be assured for such signals. To extend the procedure to the general case in which there is significant nonlinearity in the IF, one should replace the WVD in this scheme by the generalized WVD which is optimal for nonlinear FM signals [3]–[5], [8], [9]. However, good results are achieved by adjusting the window length in the XWVD method so that the IF variation is close to linear within the window. A formula for optimally adjusting the window length is [1]

$$v_{\text{opt}} = 24^{1/3} \left[\frac{d^2 f(t)}{dt^2} \right]^{-1/3}. \quad (15)$$

If good starting estimates of the IF can be obtained, this formula may be used; otherwise one may need to use trial and error. It should be noted that for nonlinear frequency laws the peak of the WVD (and hence of the XWVD) may be biased [1]. This bias for

nonlinear laws introduces a complication for the XWVD based scheme. Because there is a discrepancy between the XWVD peak and the true IF, it will not be possible to converge to the true IF. However, if the window is chosen such that the law is close to linear, one will arrive at an estimate very close to the true estimate. This is illustrated in Fig. 3. The curves on Fig. 3 represent the true IF law and the XWVD estimate for a noiseless hyperbolic FM signal (number of samples = 256, window length = 96, SNR $\rightarrow \infty$, sampling rate = 1). The laws are seen to be quite close.

Simulation of the XWVD based scheme for estimating a nonlinear IF law in the presence of noise is illustrated in Fig. 4. Fig. 4(a) shows the true IF law of the nonlinear (sinusoidally modulated) FM signal. The signal was imbedded in 0-dB white Gaussian noise, and Figs. 4(b)–(d) show IF estimates based, respectively, on the direct definition, the spectrogram peak, and the XWVD. The XWVD based estimate has the lowest variance. The final XWVD estimate is shown in Fig. 4(e). Further comparisons are given in [7] and [6].

One may also extend the XWVD based IF estimation scheme to processes which exhibit significant spread about the IF. We will refer to such a signal as a wide-band IF process. The XWVD formed from the observed signal and the synthesized reference can then give a good representation of the spread of the process about its IF. Of course, the more broadly the IF is spread, the lower the SNR will be at which the IF may be effectively estimated. Fig. 5(a) and (b) show the XWVD estimates for a “narrow-band” IF and “broad-band” IF time-varying process. Both signals were imbedded in 6-dB white Gaussian noise. The spread of the IF is well displayed in both representations.

It should be noted that the iterative XWVD based approach could also be extended to applications other than IF estimation. For example, an independent development of iterative XWVD based techniques has been applied successfully to signal synthesis [10].

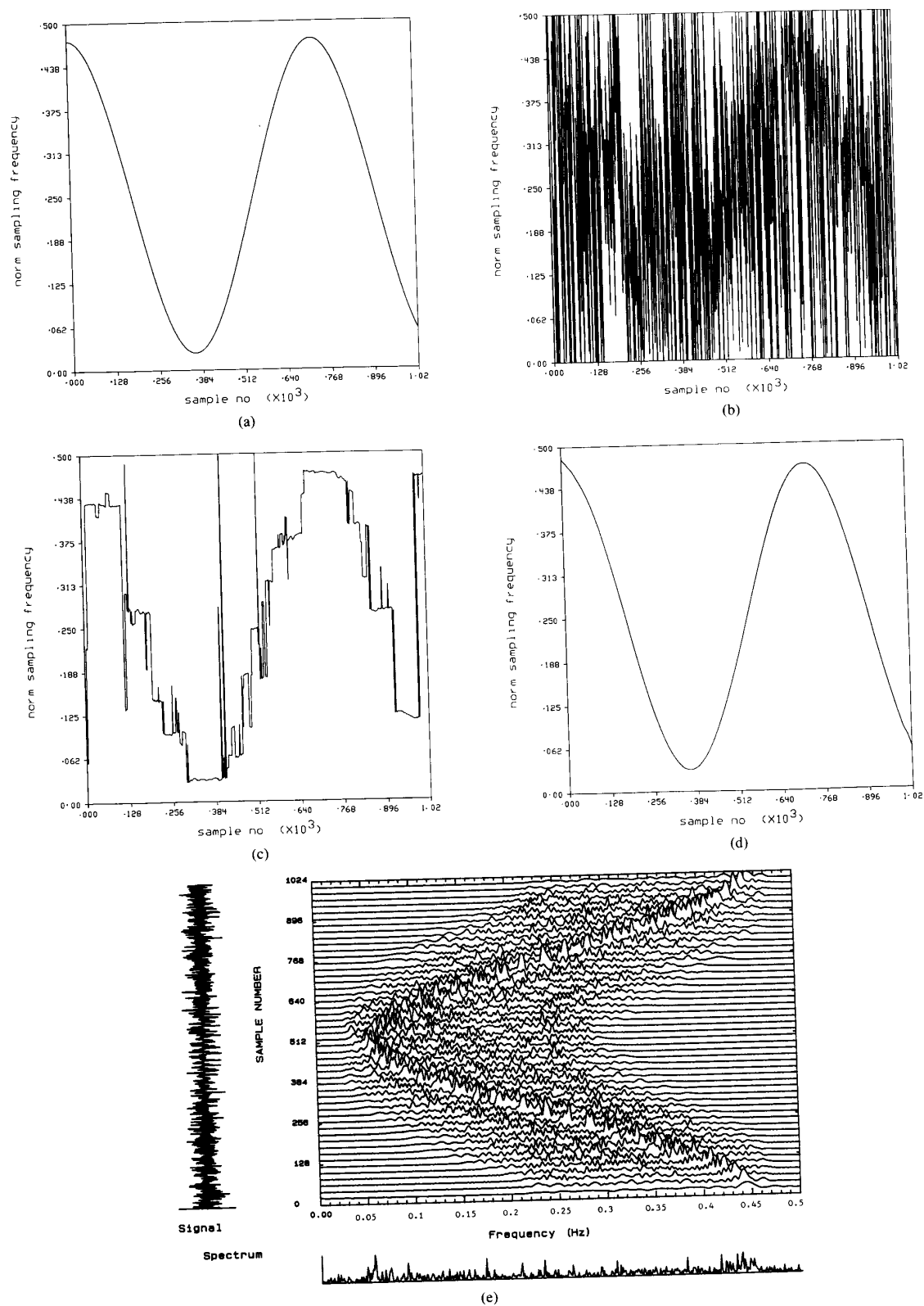


Fig. 4. (a) True IF law; (b) Direct definition IF estimate; (c) STFT peak IF estimate; (d) XWVD peak IF estimate; (e) Final XWVD estimate; $F_s = 200.0$ Hz. $N = 1024$. IRES = 20. Rectangular, length = 256.

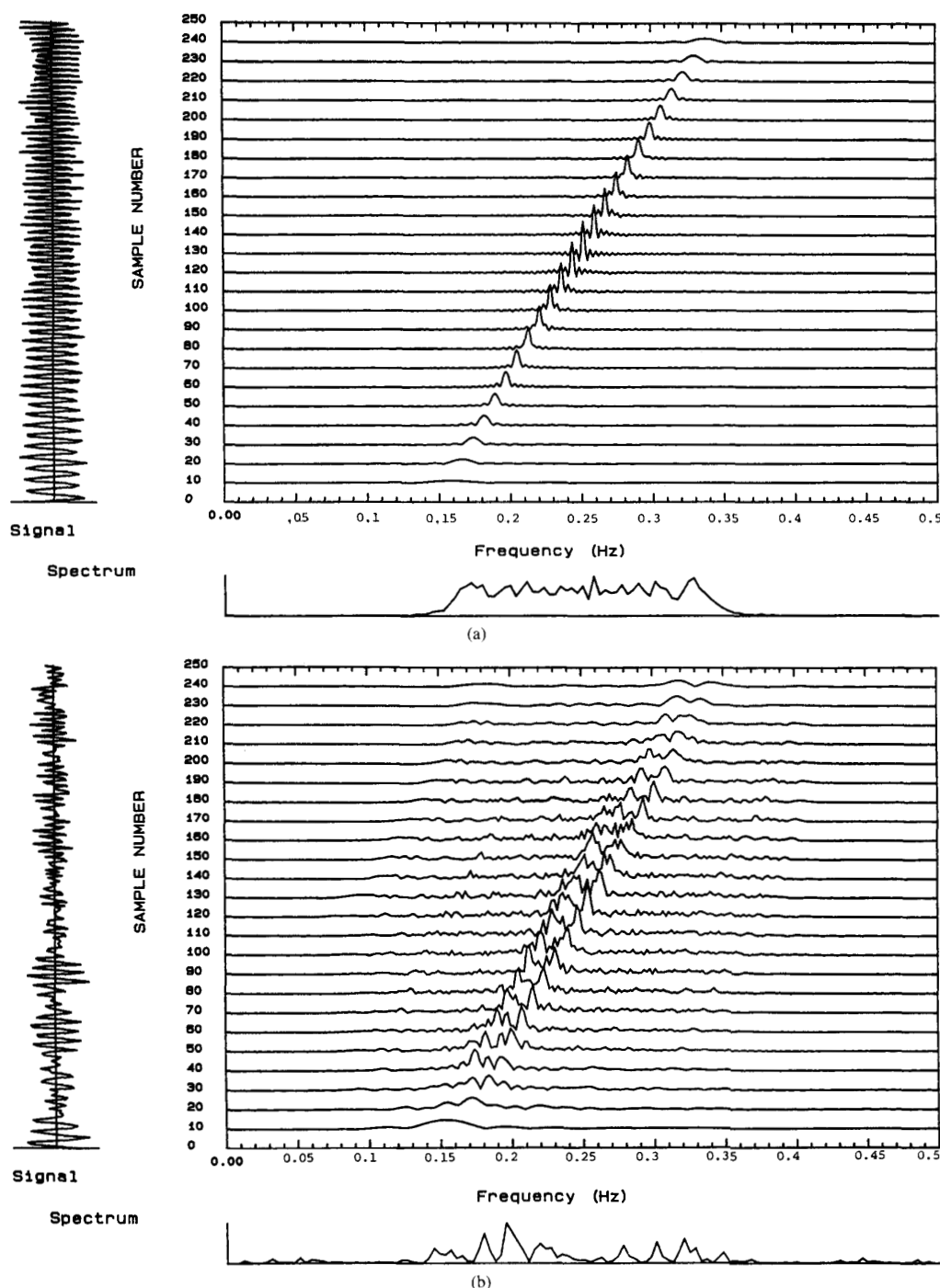


Fig. 5. (a) XWVD of narrow-band IF process; (b) XWVD of wide-band IF process. $FS = 200.0$, $N = 256$, $IRES = 10$. Rectangular, length = 256.

III. CONCLUSION

A method for IF estimation based on peak detection from the XWVD has been presented. The algorithm is iterative, with convergence being achieved for a class of practical signals within a couple of iterations. A comparison has been made with conventional methods and the XWVD based scheme has been seen to perform favorably.

REFERENCES

- [1] S. Abeysekera, "Time-frequency domain features of ECG signals: An interpretation and their application in computer aided diagnosis," Ph.D. dissertation, Univ. of Queensland, Australia, 1988.
- [2] B. Boashash, "Time-frequency signal analysis," in *Advances in Spectral Estimation and Array Processing*, vol. 1 of 2, S. Haykin, Ed. Englewood Cliffs, NJ: Prentice-Hall, 1990, ch. 9, pp. 418-517.
- [3] B. Boashash, "Interpreting and estimating the instantaneous fre-

- quency of a signal: Part I—Fundamentals, Part II—Algorithms," *Proc. IEEE*, vol. 80, no. 4, Apr. 1992.
- [4] B. Boashash and G. Frazer, "Time varying higher order spectra, generalized Wigner-Ville distributions, and the analysis of underwater acoustic data," in *Proc. Int. Conf. Acoust., Speech, Signal Processing (ICASSP)*, San Francisco, CA, Mar. 1992, pp. V-193-V196.
- [5] B. Boashash and P. J. O'Shea, "Time-varying higher order spectra," in *Proc. 9th Kobe Int. Conf. Electron. Inform. Sci.*, Kobe, Japan, June 1991.
- [6] B. Boashash, P. J. O'Shea, and M. J. Arnold, "Algorithms for instantaneous frequency estimation: A comparative study," *Proc. of SPIE Int. Soc. Opt. Eng.*, vol. 1348, pp. 24-46, Aug. 1990.
- [7] B. Boashash, P. O'Shea, and B. Ristic, "A statistical/computational comparison of some algorithms for instantaneous frequency estimation," in *ICASSP*, Toronto, 1991, pp. 3193-3196.
- [8] B. Boashash and B. Ristic, "Time varying higher order spectra," in *Proc. 25th Asilomar Conf.*, Pacific Grove, CA, Nov. 1991, 393-397.
- [9] B. Boashash and P. O'Shea, "Time-varying higher order spectra," *Proc. SPIE, Int. Soc. Opt. Eng.*, July 1991.
- [10] T. McHale and G. F. Boudreaux-Bartels, "A weighted cross Wigner distribution synthesis algorithm using partial model information," in *Proc. ASSP Workshop Spectrum Estimation Modelling*, New York, 1990.
- [11] P. Rao and F. J. Taylor, "Estimation of instantaneous frequency using the Wigner distribution," *Electron. Lett.*, vol. 26, pp. 246-248, 1990.
- [12] D. C. Rife and R. R. Boorstyn, "Single tone parameter estimation from discrete-time observations," *IEEE Trans. Inform. Theory*, vol. 20, no. 5, pp. 591-598, 1974.
- [13] J. Ville, "Theorie et application de la notion de signal analytique," *Cables et Transmissions*, vol. 2A, no. 1, pp. 61-74, 1948.

The Constrained MUSIC Problem

Ronald D. DeGroat, Eric M. Dowling, and Darel A. Linebarger

Abstract—In this correspondence, the MUSIC based direction-of-arrival (DOA) method is generalized to include constraints involving known signal information. Projection operators are used to constrain the noise subspace to be orthogonal to a set of prespecified direction vectors. By incorporating known source directions, the estimation of unknown source directions can be significantly improved. Simulations are performed over a wide range of scenarios to demonstrate the usefulness of the new approach.

I. INTRODUCTION

In the field of signal processing, MUSIC [1] has had a major impact on high resolution frequency and direction-of-arrival (DOA) estimation problems. In this correspondence, the basic concept of MUSIC is extended to include constraints involving known signal information. By incorporating known signal information, the estimation of unknown signal information can be greatly improved. Although constrained MUSIC can be applied to frequency estimation, we focus on the DOA estimation problem. In the DOA

problem, unconstrained MUSIC produces direction estimates by searching the array manifold for direction vectors that are orthogonal or nearly orthogonal to a "noise" subspace. In the constrained MUSIC problem, we use projection operators to constrain the noise subspace to be orthogonal to a set of prespecified constraint vectors. The constraint vectors typically represent known source directions. However, the constraints are not limited to source directions. In certain applications, other constraints and projections may be appropriate. It should be emphasized, however, that the constraints described in this correspondence are suitable for problems in which some of the source directions are known *a priori*. In other words, accurate knowledge of certain source directions is assumed to be available. This kind of situation might arise, for example, if echoes are received from a tower (or other fixed structure) that lies in the viewing field of an active radar.

To simplify the initial presentation of constrained MUSIC, we will assume that q complex monochromatic (possibly coherent) plane waves are incident on an array of m sensors in a nondispersive medium. With additive sensor noise, the k th snapshot vector can be written as

$$\mathbf{x}(k) = \mathbf{A}s(k) + \mathbf{n}(k) \quad (1)$$

where $\mathbf{A} = [\mathbf{a}(\theta_1) \cdots \mathbf{a}(\theta_q)]_{m \times q}$ is a matrix of source direction vectors, $s(k) = e^{j\omega k} [\alpha_1(k), \cdots, \alpha_q(k)]^T$ is a vector of q monochromatic signals, $\alpha_i(k) e^{j\omega k}$, $i = 1 \cdots q$, for which the $\alpha_i(k)$ are random complex amplitudes, and the elements of the noise vector, $\mathbf{n}(k) = [n_1(k), \cdots, n_m(k)]^T$ consist of zero mean, white, complex Gaussian noise. It is assumed that the signal and noise are uncorrelated with each other. In the case of a linear equispaced array, the direction vectors are defined by the array manifold $\mathbf{a}(\theta) = [1, e^{j\pi d \cos \theta}, e^{j2\pi d \cos \theta}, \cdots, e^{j(m-1)\pi d \cos \theta}]^T$ where d is the sensor spacing in units of half wavelengths ($\lambda = \omega/2\pi c$, c = wave propagation speed) and θ is the direction angle measured with respect to the main axis of the array. For the data model in (1), the correlation matrix is of the form

$$\begin{aligned} \mathbf{R} &= E[\mathbf{x}_k \mathbf{x}_k^H] \quad \forall k \\ &= \mathbf{A} \mathbf{P} \mathbf{A}^H + \sigma^2 \mathbf{I} \end{aligned} \quad (2)$$

where $\mathbf{P} = E[s_k s_k^H]$, $\forall k$ is the $q \times q$ signal correlation matrix, σ^2 is the white noise power (or variance) and $E[\cdot]$ is the expectation operator. Coherence between the i th and j th signals is defined by

$$\rho_{ij} = \frac{E[s_i(k) s_j^*(k)]}{\sqrt{E[s_i^2(k)] E[s_j^2(k)]}} \quad (3)$$

Based on the model in (1), let

$$\mathbf{X} = [\mathbf{x}_1 | \mathbf{x}_2 | \cdots | \mathbf{x}_n]_{m \times n} \quad (4)$$

be a full rank data matrix containing n snapshot vectors. It is assumed that $n \geq m > q$. The noise subspace used in MUSIC can be obtained from the singular value decomposition (SVD) of \mathbf{X} (or from the eigenvalue decomposition of $\hat{\mathbf{R}} = (1/n) \mathbf{X} \mathbf{X}^H$, but for numerical reasons, the SVD approach is preferred). Suppose the SVD of \mathbf{X} is given by

$$\begin{aligned} \mathbf{X} &= [\mathbf{U}_1 | \mathbf{U}_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^H \\ \mathbf{V}_2^H \end{bmatrix} \\ &= \underbrace{\mathbf{U}_1}_{m \times q} \underbrace{\Sigma_1}_{q \times q} \underbrace{\mathbf{V}_1^H}_{q \times n} + \underbrace{\mathbf{U}_2}_{m \times s} \underbrace{\Sigma_2}_{s \times s} \underbrace{\mathbf{V}_2^H}_{s \times n} \end{aligned} \quad (5)$$

Manuscript received June 9, 1991; revised March 23, 1992. This work was supported in part by the Texas Advanced Technology Program Grant 009741-038, the Texas Advanced Research Program Grant 009741-039, and Texas Instruments.

The authors are with the Erik Jonsson School of Engineering and Computer Science, University of Texas at Dallas, Richardson, TX 75083-0688. IEEE Log Number 9206005.